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Alternation Properties of Tchebyshev-Systems and the Existence of Adjoined Functions

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A finite-dimensional linear space of functions is called a T-space (Tchebyshev-space) iff it has a basis satisfying Haar's condition. A function f is called adjoined to an n-dimensional T-space U_n iff span $U_n \cup \{f\}$ is an $(n + 1)$-dimensional T-space.

Rutman mentioned (see Krein [2]) that there are T-spaces for which no adjoined functions exist. Apparently, no such example has been published.

Laasonen [3] showed that if U_n consists of n times continuously differentiable functions defined on an interval, then there is a function adjoined to U_n. Later Karlin and Studden [1] proved the same and then applied a rather complicated limiting process to infer the same conclusion when U_n consists of continuous functions defined on an interval.

Rutman [4] stated that if U_n consists of functions continuous from the right and defined on an open interval, there is a function adjoined to U_n. Unfortunately, he gave only an outline of his proof.

Throughout this paper we shall consider only T-spaces of functions defined on totally ordered sets. We need the following definitions.

DEFINITION. A totally ordered set M has property (D) if it contains no smallest or greatest element and for every two distinct elements of M there is an element between them.

DEFINITION. Let U_n be an n-dimensional T-space (of functions defined) on a totally ordered set M. U_n is called oriented iff for every $f \in U_n$ there are at most n points $t_1, ..., t_n \in M$ with $t_1 < \cdots < t_n$ and sign $f(t_i) = -\text{sign} f(t_{i+1}) \neq 0$ for $i = 1, ..., n - 1$.

In a previous paper [5] we proved that if M is a totally ordered set and has property (D), and U_n is an n-dimensional oriented T-space on M, $n \geqslant 2$, U_n contains an $(n - 1)$-dimensional oriented T-space. Our main result is that under the same assumptions there is a function adjoined to U_n such that span $U_n \cup \{f\}$ is oriented, too.

The proof consists of two parts. In Section 2 we define an operation "relative differentiation" that transforms U_n into an $(n-1)$-dimensional T-space. In the second part of the proof we show that if there is a function adjoined to this space, the same is true for U_n.

Considerable simplification has been obtained in many arguments by replacing determinant inequalities by alternation properties. In addition, no limit process (as in [1, p. 241–246]) is needed.

Some of the results, such as Theorems 1 and 2, seem to be of independent interest.

1. Preliminaries, Chains of T-Spaces

We first recall a few definitions and propositions from [5].

Definition. Let M be a totally ordered set, f a real-valued function defined on M, and $t_1,..., t_k \in M$ with $t_1 < \cdots < t_k$.

(a) $t_1,..., t_k$ form a strong alternation of f of length k iff $\operatorname{sign} f(t_i) = -\operatorname{sign} f(t_{i+1}) \neq 0$ for $i = 1,..., k-1$.

(b) $t_1,..., t_k$ form a weak alternation of f of length k iff $\operatorname{sign} f(t_i) = -\operatorname{sign} f(t_{i+1})$ for $i = 1,..., k-1$.

(c) $t_1,..., t_n$ form a quasialternation of f of length k iff $\operatorname{sign}(f(t_i) - f(t_{i+1})) = -\operatorname{sign}(f(t_{i+1}) - f(t_{i+2}))$ for $i = 1,..., k-2$.

Lemma 1. *Let M be a totally ordered set, and U_n an n-dimensional linear space of functions defined on M. Then the following statements are equivalent*:

(a) *U_n is an oriented T-space.*

(b) *U_n is a T-space, and no $f \in U_n \backslash \{0\}$ has a weak alternation of length exceeding n.*

(c) *If $f_1,..., f_n$ is a basis of M, $\det(f_i(t_j))_{n,n}$ has constant sign for all $t_1,..., t_n \in M$ with $t_1 < \cdots < t_n$.*

Lemma 2. *Let M be a totally ordered set, and U_n an n-dimensional oriented T-space on M, $n \geqslant 1$. Assume $f \in U_n \backslash \{0\}$ with zeros $t_1 < \cdots < t_{n-1}$. Then all point sets $s_1,..., s_n \in M$ with $s_1 < t_1 < s_2 < t_2 < \cdots < t_{n-1} < s_n$ form strong alternations of f.*

For the following it will be convenient to define chains of T-spaces.

DEFINITION. The T-spaces U_i, $\dim U_i = i$, $i = 1,..., n$, form a chain, if $U_1 \subset \cdots \subset U_n$.

If, moreover, U_1 consists of the constant functions, the chain is called normed.

In the following we shall use a stronger version of the concept of adjoined functions.

DEFINITION. Let U_n be an n-dimensional oriented T-space on a totally ordered set M. f is called strongly adjoined to U_n iff span $U_n \cup \{f\}$ is an $(n+1)$-dimensional oriented T-space on M.

If U_n is an n-dimensional oriented T-space on a set M which has property (D), we may by Corollary 2 in [5] assume that there is a chain $U_1 \subset \cdots \subset U_n$ of oriented i-dimensional T-spaces U_i, $i = 1,..., n$.

For the proof of existence of a strongly adjoined function under the above hypotheses, we may further assume that the chain $U_1 \subset \cdots \subset U_n$ is normed.

In the following we mean by hypothesis (A): M is totally ordered, $U_1 \subset \cdots \subset U_n$ is a normed chain of oriented T-spaces on M with $n \geqslant 2$.

LEMMA 3. *If hypothesis (A) is fulfilled, and $f \in U_n \backslash U_{n-1}$, every quasi-alternation of f has at most length n.*

Proof. $n = 2$. Let $f \in U_2 \backslash U_1$ and $g(t) = 1$ for $t \in M$. Then f and g form a basis of U_2, and for $t, u \in M$ with $t < u$

$$\det \begin{pmatrix} 1 & 1 \\ f(t) & f(u) \end{pmatrix} = f(u) - f(t)$$

has constant sign $\neq 0$ because U_2 is oriented. So f is strictly monotonous.

$n - 1 \Rightarrow n$. Suppose there is an $f \in U_n \backslash U_{n-1}$ and points $t_1 < \cdots < t_{n+1}$ with

$$f(t_i) \geqslant f(t_{i+1}), \qquad \text{if } i \text{ is odd},$$

$$f(t_i) \leqslant f(t_{i+1}), \qquad \text{if } i \text{ is even}, \qquad i = 1,..., n.$$

Let $g \in U_{n-1}$ be the function that interpolates f in $t_2,..., t_n$. By induction hypothesis we have

$$g(t) \leqslant g(t_2) \qquad \text{for} \quad t < t_2,$$

$$g(t) \leqslant g(t_n) \qquad \text{for} \quad t > t_n, \qquad \text{if } n \text{ is even},$$

$$g(t) \geqslant g(t_n) \qquad \text{for} \quad t > t_n, \qquad \text{if } n \text{ is odd}.$$

Hence, $t_1, \ldots, t_{n+1}$ is a weak alternation of $f - g$ of length $n + 1$, contradicting the hypothesis that U_n is oriented.

Remarks. The statement of Lemma 3 is false if U_n contains no T-space of dimension $n - 1$. In [5, Example 3], R is a three-dimensional oriented T-space containing no two-dimensional T-subspace. Indeed, for sufficiently small $\epsilon > 0$ the function $f_\epsilon \in R$ with $f_\epsilon(t) := t \sin(t - \epsilon)$ has quasialternations of length 4.

Lemma 3 says that for every $f \in U_n \backslash U_{n-1}$ there are at most $n - 2$ points $t_1, \ldots, t_{n-2} \in M$ with $t_1 < \cdots < t_{n-2}$ such that for the sets $A_1 := \{x \in M \mid x < t_1\}$, $A_k := \{x \in M \mid t_{k-1} < x < t_k\}$, $k = 2, \ldots, n - 2$, $A_{n-1} := \{x \in M \mid t_{n-2} < x\}$ either f or $-f$ is strictly increasing on $A_1, A_3, A_5, \ldots$ and strictly decreasing on $A_2, A_4, A_6, \ldots$.

DEFINITION. If M is totally ordered, we define $I(a, b) := \{x \in M \mid a \leqslant x \leqslant b\}$ for all $a, b \in M$ with $a < b$.

LEMMA 4. *If hypothesis (A) is fulfilled, and $f \in U_n$, then f is bounded on every set $I(a, b)$.*

Proof. $n = 2$. As every $f \in U_2$ is strictly monotonous on M, on $I(a, b)$ the function f is bounded by $f(a)$ and $f(b)$.

$n - 1 \Rightarrow n$. Suppose there is $f \in U_n \backslash U_{n-1}$ and $a, b \in M$ with $a < b$ such that f is not bounded on $I(a, b)$. Because of Lemma 3 there are $c, d \in I(a, b)$ with $c < d$ such that f is strictly monotonous and unbounded on $I(c, d) \backslash \{d\}$. Without loss of generality let f be monotonously increasing and unbounded from above on $I(c, d)$. Obviously, $I(c, d)$ contains infinitely many points.

Let $t_1, \ldots, t_{n-1} \in I(c, d)$ with $t_1 < \cdots < t_{n-1} < d$. Without loss of generality assume $f(t_{n-1}) = 0$. Define $g \in U_{n-1}$ by

$$g(t_i) = (-1)^{n-1-i} \quad \text{for} \quad i = 1, \ldots, n - 2, \qquad g(t_{n-1}) = 0. \tag{1}$$

For $t > t_{n-1}$ we have $g(t) > 0$, because otherwise g had a weak alternation of length n. By induction hypothesis g is bounded on $I(a, b)$, say $|g(t)| < K$ for $t \in I(a, b)$.

Let $t_{n+1} = b$. Then there is an $\alpha > 0$ such that $t_1, \ldots, t_{n-1}$ is a weak alternation of $g - \alpha f$ and

$$(g - \alpha f)(t_{n-2}) < 0, \qquad (g - \alpha f)(t_{n-1}) = 0, \qquad (g - \alpha f)(t_{n+1}) > 0. \tag{2}$$

As f is unbounded from above on $I(t_{n-1}, t_{n+1})$, there is a $t_n \in M$ with $t_{n-1} < t_n < t_{n+1}$ and $(g - \alpha f)(t_n) < 0$. Thus, $t_1, \ldots, t_{n+1}$ is a weak alternation of $g - \alpha f$ of length $n + 1$, contradicting the hypothesis that U_n is oriented.

As an application we get the following theorem.

THEOREM 1. *Let $M = (a, b)$ be an open interval and $U_1 \subset \cdots \subset U_n \subset C(M)$ a normed chain of T-spaces. If all $f \in U_n$ are bounded, they may be continuously extended to functions $\bar{f}$ defined on $\bar{M}$, and the spaces $\bar{U}_i := \{\bar{f} \in C(\bar{M}) \mid \bar{f}|_M = f \text{ for an } f \in U_i\}$ form a normed chain of T-spaces on $\bar{M}$.*

Proof. Suppose there is an $f \in U_i \backslash \{0\}$ such that its extension $\bar{f}$ has i zeros $t_1 < \cdots < t_i$. By Lemma 3 M can be split into at most $i - 1$ intervals in each of which f is strictly monotonous. The first interval lies left of t_2, and the second contains points left of t_2, too. Part of the third interval lies left of t_3, and so on until finally part of the $(i - 1)$st interval lies left of t_{i-1}. So f is strictly monotonous right of t_{i-1} and cannot approach zero again.

2. RELATIVE DERIVATIVES

DEFINITION. Let M be totally ordered, $a \in M$, and f a real-valued function defined on M. Then f has the right side limit α in a (written $\alpha = M - \lim_{x \to a+} f(x)$), iff for every $\epsilon > 0$ there is a $y \in M$ with $a < y$ and $|f(x) - \alpha| < \epsilon$ for all $x \in M$ with $a < x \leqslant y$.

The limits $M - \lim_{x \to a+} \inf f(x)$ and $M - \lim_{x \to a+} \sup f(x)$ and the corresponding left side limits are defined analogously. If, for example, $a, b \in M$ are two points with $a < b$ such that there is no point between them, we have $M - \lim_{x \to a+} f(x) = f(b)$ and $M - \lim_{x \to b-} f(x) = f(a)$.

THEOREM 2. *Let hypothesis (A) be satisfied, and assume M contains no smallest or greatest element. Then we have for any fixed $f_2 \in U_2 \backslash U_1$:*

(a) *There is a linear operator $D_+ \colon U_n \to \mathbb{R}^n$, defined by*

$$(D_+ f)(a) := M - \lim_{x \to a+} \frac{f(x) - f(a)}{f_2(x) - f_2(a)}.$$

(b) *The spaces $D_+ U_i := \{D_+ f \mid f \in U_i\}$, $i = 1,..., n$, have dimension $i - 1$ and form a normed chain of oriented T-spaces on M.*

Proof. $n = 2$. $D_+ U_2$ is the space of constant functions on M.

$n - 1 \Rightarrow n$: (1) *Existence of $D_+ f$.* Let $a \in M$ fixed, and $f \in U_n$. If there is a $y \in M$ with $a < y$ such that no point of M lies between a and y, we have

$D_+f(a) = (f(y) - f(a)/(f_2(y) - f_2(a))$. Now we assume that for every $y \in M$ with $a < y$ there is an $x \in M$ with $a < x < y$. Without loss of generality we further assume that $f_2(a) = f(a) = 0$, f_2 is strictly increasing and there is a $y_1 \in M$ with $a < y_1$ such that for all $t \in M$ with $a < t < y_1$ we have $f(t) > 0$.

First suppose

$$M - \lim_{t\to a+} \sup \frac{f(t)}{f_2(t)} = \infty.$$

Let $t_1, \ldots, t_{n-1} \in M$ with $t_1 < \cdots < t_{n-1} = a$, and let $g \in U_{n-1}$ be defined by (1). $g(t)$ is positive for $t > a$, because otherwise g would have a weak alternation of length n. As $f(t)/f_2(t) = (f(t)/g(t)/(g(t)/f_2(t))$ for $t > a$ and $D_+g(a) = M - \lim_{t\to a+}(g(t)/f_2(t))$ exists by induction hypothesis, it follows that $M - \lim_{t\to a+} \sup(f(t)/g(t)) = \infty$.

Let $t_{n+1} \in M$ with $t_{n+1} > a$. Then there is an $\alpha > 0$ such that $t_1, \ldots, t_{n-1}$ is a weak alternation of $g - \alpha f$ with (2). Because of $M - \lim_{t\to a+} \sup(f(t)/g(t)) = \infty$ there is a $t_n \in M$ with $t_{n-1} < t_n < t_{n+1}$ and $(g - \alpha f)(t_n) < 0$. The points $t_1, \ldots, t_{n+1}$ form a weak alternation of $g - \alpha f$ of length $n + 1$ in contradiction to Lemma 1.

Now suppose we had

$$\beta := M - \lim_{t\to a+} \inf (f(t)/f_2(t)) < \gamma := M - \lim_{t\to a+} \sup (f(t)/f_2(t)).$$

If we let $\delta := (\beta + \gamma)/2$, then for all $y \in M$ with $a < y$ there are $u, v \in M$ with $a < u$, $v < y$ and $(f - \delta f_2)(u) < 0 < (f - \delta f_2)(v)$. As we may choose $y = u$ or $y = v$, there exist sequences $u_1, u_3, \ldots$ and $v_2, v_4, \ldots$ in M with $u_1 > v_2 > u_3 > v_4 > \cdots > a$ and $(f - \delta f_2)(u_i) < 0 < (f - \delta f_2)(v_{i+1})$ for $i = 1, 3, 5, \ldots$. This again contradicts Lemma 1.

(2) *D_+U_n is an $(n - 1)$-dimensional T-space.* Suppose there is an $f \in U_n \backslash U_1$ such that D_+f has zeros $t_1 < \cdots < t_{n-1}$. Let $g \in U_{n-1}$ be the function that interpolates f in $t_1, \ldots, t_{n-1}$. Then we have $D_+(g - f)(t_i) = D_+g(t_i)$ for $i = 1, \ldots, n - 1$, and $D_+g \in D_+U_{n-1}$. From Lemma 2 it follows that $t_1, \ldots, t_{n-1}$ is a weak alternation of $D_+(g - f)$ and so of D_+g, too. This contradicts the induction hypothesis that D_+U_{n-1} is an $(n - 2)$-dimensional oriented T-space. Because of $U_1 = \text{kernel } D_+$ we have

$$\dim D_+U_n = \dim U_n - \dim \text{kernel } D_+ = n - 1.$$

(3) *D_+U_n is oriented.* Suppose there are $f \in U_n$ and points $t_1, \ldots, t_n \in M$ with $t_1 < \cdots < t_n$ and $(-1)^i D_+f(t_i) > 0$ for $i = 1, \ldots, n$. Then there are points $u_1, \ldots, u_n \in M$ with $t_1 < u_1 \leqslant t_2 < \cdots \leqslant t_n < u_n$ and $(-1)^i (f(u_i) - f(t_i))/(f_2(u_i) - f_2(t_i)) > 0$ for $i = 1, \ldots, n$. So we have $(-1)^i (f(u_i) - f(t_i)) > 0$ for $i = 1, \ldots, n$. This, however, contradicts Lemma 3.

Remark. Theorem 2 does not hold if the T-spaces are not oriented, as the following example shows. Let

$$M = (-1, 1), f_1(t) = 1,$$
$$f_2(t) = t,$$
$$f_3(t) = \begin{cases} t^2 - 1 & \text{for } t \in (0, 1) \\ 1 - t^2 & \text{for } t \in (-1, 0], \end{cases}$$

and

$$U_i = \text{span}\{f_1, ..., f_i\}, \qquad i = 1, 2, 3.$$

$U_1 \subset U_2 \subset U_3$ is a normed chain of T-spaces, but U_3 is not oriented, and we have

$$M - \lim_{x \to 0+} \frac{f_3(x) - f_3(0)}{f_2(x) - f_2(0)} = \infty.$$

The version of Theorem 2 for left side limits is obtained by replacing "D_+" by "D_-" and "$M - \lim_{x \to a+}$" by "$M - \lim_{x \to a-}$" in Theorem 2.

3. Real Domains, Adjoined Functions

With hypothesis (A) every $f_2 \in U_2 \backslash U_1$ is strictly monotonous on M by Lemma 3. We define $V_i := \{h: f_2(M) \to \mathbb{R} \mid h = f \circ f_2^{-1} \text{ for an } f \in U_i\}$, $i = 1, ..., n$. Then $V_1 \subset \cdots \subset V_n$ is a normed chain of oriented T-spaces on $f_2(M)$, and V_2 consists of the linear functions restricted to $f_2(M)$. The following statements are obvious.

(1) If M has property (D), so does $f_2(M)$.

(2) If M is a real interval and f_2 is continuous, $f_2(M)$ is an interval, and f_2 is a homeomorphism.

(3) A function $g: f_2(M) \to \mathbb{R}$ is strongly adjoined to V_n iff $g \circ f_2$ is strongly adjoined to U_n.

For the proof of existence of strongly adjoined functions under hypothesis (A), we may—because of the last statement—assume that the domain M is real and that U_2 consists of the linear functions restricted to M.

LEMMA 5. *Assume that hypothesis* (A) *is fulfilled, that M is real, that M contains no smallest or greatest element and that U_2 consists of the linear functions restricted to M. Denote by I the open interval* (inf M, sup M). *Then every $f \in U_n$ can be extended to a function $\bar{f}$ continuous on I.*

Proof. $n = 2$. Every $f \in U_2$ can be extended to a linear function on I.

$n - 1 \Rightarrow n$. First we show that every $f \in U_n$ may be continuously extended to $I \cap \overline{M}$. Let $a \in I \cap (\overline{M} \backslash M)$ and $f \in U_n$ fixed. Because of Lemma 3, I can be split into $k \leqslant n - 1$ subintervals $A_1, \ldots, A_k$ such that f is strictly monotonous on each of the sets $M \cap A_i$, $i = 1, \ldots, k$. As M contains no smallest or greatest element, there exist $b := \sup\{x \in M \mid x < a\}$ and $c := \inf\{x \in M \mid a < x\}$. Because of Lemmas 3 and 4 there exist the limits

$$f_-(b) := M - \lim_{x \to b-} f(x) \qquad \text{and} \quad f_+(c) := M - \lim_{x \to c+} f(x).$$

For $b = a < c$ we define $\bar{f}(a) = f_-(b)$, for $b < a = c$ we define $\bar{f}(a) = f_+(c)$. For $b = a = c$ it remains to show: $f_-(a) = f_+(a)$.

Suppose we had $f_-(a) \neq f_+(a)$, say $0 = f_-(a) < f_+(a)$ without loss of generality. Let $t_1, \ldots, t_{n-2} \in M$ with $t_1 < \cdots < t_{n-2} < a$. For every $h \in U_{n-1} \backslash \{0\}$ with $h(t_1) = \cdots = h(t_{n-2}) = 0$ we have $\bar{h}(a) \neq 0$ because of Lemma 3. Now let $h_0 \in U_{n-1}$ with $h_0(t_i) = (-1)^{n-1-i}$ for $i = 1, \ldots, n-2$. For $g := \bar{h}(a) \cdot h_0 - \bar{h}_0(a) \cdot h$ we get $g(t_i) = (-1)^{n-1-i}$ for $i = 1, \ldots, n - 2$ and $\bar{g}(a) = 0$. Furthermore, Lemma 3 yields $g(t) > 0$ for all $t \in M$ with $a < t$. Let $t_{n+1} \in M$ with $a < t_{n+1}$. Then there is an $\alpha > 0$ such that $t_1, \ldots, t_{n-2}$ form a strong alternation of $g - \alpha f$ and

$$(g - \alpha f)(t_{n-2}) < 0,$$
$$(\bar{g} - \alpha f)(a) = 0,$$
$$(g - \alpha f)(t_{n+1}) > 0.$$

As we have $f_+(a) > \bar{g}(a) = 0$, there is a $t_n \in M$ with $a < t_n < t_{n+1}$ and $(g - \alpha f)(t_n) < 0$. Besides, there is a $t_{n-1} \in M$ with $t_{n-2} < t_{n-1} < a$ such that $t_1, \ldots, t_{n+1}$ is a quasialternation of $g - \alpha f$ of length $n + 1$, contradicting Lemma 3. For $t \in I \backslash \overline{M}$ we put

$$\bar{f}(t) = \bar{f}(t_-) + \frac{t - t_-}{t_+ - t_-} (\bar{f}(t_+) - \bar{f}(t_-)),$$

where $t_- = \max\{u \in \overline{M} \mid u < t\}$ and $t_+ = \min\{u \in \overline{M} \mid u > t\}$. Thereby we fill the "gaps" of $\overline{M}$ with linear functions.

DEFINITION. Under the hypotheses of Lemma 5 we denote by $\overline{U}_i$ the space of the functions $\bar{f}$ with $f \in U_i$ as constructed in the proof of Lemma 5.

LEMMA 6. *If the hypotheses of Lemma 5 are fulfilled and A is a compact subinterval of I, for every fixed $\bar{f} \in \overline{U}_n$ the difference quotient $(\bar{f}(b) - \bar{f}(a))/(b - a)$, $a, b \in A$, $a \neq b$, is bounded.*

Proof. $n = 2$. The difference quotient is a constant.

$n-1 \Rightarrow n$. Suppose there are sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in $A \cap M$ and an $f \in U_n$ with $a_k < b_k$ for all k and $|(f(b_k) - f(a_k))/(b_k - a_k)| \to \infty$ for $k \to \infty$. As A is compact we may without loss of generality assume that there are points $a, b \in A \cap \overline{M}$ with $a_k \to a$ and $b_k \to b$ for $k \to \infty$. As $|f(b_k) - f(a_k)|$ is bounded because of Lemma 4, $b_k - a_k$ goes to zero for $k \to \infty$, and so we have $a = b$.

Without loss of generality let $\bar{f}(a) = 0$ and $(f(b_k) - f(a_k))/(b_k - a_k) \to \infty$ for $k \to \infty$. Let $t_1, \ldots, t_{n-1} \in M$ with $t_1 < \cdots < t_{n-1} < a$, and define $g \in U_{n-1}$ by (1). g is strictly increasing on $M \cap (t_{n-1}, \infty)$ because of Lemma 3. Choose $t_{n+1} \in M$ with $a < t_{n+1}$. There is an $\alpha > 0$ such that $t_1, \ldots, t_{n-2}$ is a strong alternation of $g - \alpha f$ and $(g - \alpha f)(t_{n-2}) < (\bar{g} - \alpha\bar{f})(a) < (g - \alpha f)(t_{n+1})$. If k is sufficiently large, we get $t_{n-2} < a_k < b_k < t_{n+1}$,

$$(g - \alpha f)(t_{n-2}) < \min\{(g - \alpha f)(a_k), (g - \alpha f)(b_k)\}$$
$$< \max\{(g - \alpha f)(a_k), (g - \alpha f)(b_k)\} < (g - \alpha f)(t_{n+1}),$$

and from the induction hypothesis

$$\frac{(g - \alpha f)(b_k) - (g - \alpha f)(a_k)}{b_k - a_k} = \frac{g(b_k) - g(a_k)}{b_k - a_k} - \alpha \frac{f(b_k) - f(a_k)}{b_k - a_k} < 0.$$

Hence follows $(g - \alpha f)(a_k) > (g - \alpha f)(b_k)$, and $t_1, \ldots, t_{n-2}, a_k, b_k, t_{n+1}$ form a quasialternation of $g - \alpha f$ of length $n + 1$, contradicting Lemma 3. Thus, the difference quotient $(f(y) - f(x))/(y - x)$ is bounded for $x, y \in A \cap M$, $x \neq y$, and so for $x, y \in A \cap \overline{M}$, $x \neq y$, too.

Now choose $x \in A \backslash \overline{M}$ and $y \in A$. Then x is an inner point of an interval $[1, r]$ with $1, r \in A \cap \overline{M}$, on which $\bar{f}$ is a linear function. For $y \in [1, r]$ we have $(f(y) - f(x))/(y - x) = (f(r) - f(1))/(r - 1)$. For $y \notin [1, r]$, it is easy to see that

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leqslant \max \left\{ \left| \frac{f(y) - f(1)}{y - 1} \right|, \left| \frac{f(y) - f(r)}{y - r} \right|, \left| \frac{f(r) - f(1)}{r - 1} \right| \right\}.$$

For $y \in A \cap \overline{M}$ the right side is bounded, for $y \notin A \cap \overline{M}$ replace (x, y) by $(y, 1)$ or (y, r) and apply the argument again.

LEMMA 7. *Under the hypotheses of Lemma 5 and with the above notations every $\bar{f} \in \overline{U}_n$ has the following properties*:

(a) *$\bar{f}$ is absolutely continuous on every compact subinterval of I.*

(b) *$\bar{f}$ is right side differentiable on I.*

(c) *The right side derivative* $\bar{f}_+'$ *has no strong atlernations of length greater* $n - 1$.

(d) $\bar{f}_+'$ *is bounded on every compact subinterval of* I.

(e) $\bar{f}_+'$ *is right side continuous on* I.

Proof. Let $\bar{f} \in \bar{U}_n$ fixed:

(a) By Lemma 6 $\bar{f}$ is Lipschitz-bounded on every compact subinterval of I and thus absolutely continuous.

(b) From Lemma 6 it is clear that for every $a \in I$ we have

$$\lim_{x \to a+} \sup \frac{\bar{f}(x) - \bar{f}(a)}{x - a} < \infty.$$

Without loss of generality let $a = \bar{f}(a) = 0$.

Suppose $\beta := \lim_{x \to 0+} \inf(\bar{f}(x)/x) < \gamma := \lim_{x \to 0+} \sup(\bar{f}(x)/x)$. The construction of $\bar{f}$ then yields $0 \in \bar{M}$ and

$$M - \lim_{x \to 0+} \inf(f(x)/x) < M - \lim_{x \to 0+} \sup(f(x)/x),$$

which is led to a contradiction in the same way as in part (1) of the proof of Theorem 2.

(c) Suppose $\bar{f}_+'$ has a strong alternation of length n. In the same way as in part (3) of the proof of Theorem 2 it is shown that then there is an $f \in U_n$ with a quasialternation of length $n + 1$, contradicting Lemma 3.

(d) The statement is clear because $\bar{f}$ is Lipschitz-bounded on every compact subinterval of I.

(e) Let $a \in I$. If a lies in an interval of $I \backslash \bar{M}$ or is the left endpoint of such an interval, the statement follows from the construction of $\bar{f}$. If a is the limit of a decreasing sequence of points in M, let $a = \bar{f}(a) = 0$ without loss of generality. Because of Lemma 7(c) and (d) there is an $\epsilon > 0$ such that $\bar{f}_+'$ is monotonous and bounded in $[0, \epsilon]$, say monotonously increasing. Then $\bar{f}$ is convex in $[0, \epsilon]$, and for all $x, y \in (0, \epsilon)$ with $x < y < 2x$ we have

$$\frac{\bar{f}(y) - \bar{f}(x)}{y - x} \leqslant \frac{\bar{f}(2x) - \bar{f}(x)}{x} = 2\,\frac{\bar{f}(2x)}{2x} - \frac{\bar{f}(x)}{x},$$

and so for every $x \in (0, \epsilon/2)$

$$\bar{f}_+'(x) < 2\,\frac{\bar{f}(2x)}{2x} - \frac{\bar{f}(x)}{x}.$$

Hence, follows $\lim_{x \to 0+} \bar{f}_+'(x) \leqslant \bar{f}_+'(0)$, and so $\bar{f}_+'(0) = \lim_{x \to 0+} \bar{f}_+'(x)$.

DEFINITION. If U is a linear space of functions defined on a set M, for every subset N of M we denote by $E_N{}^M$ the projection operator defined by $E_N{}^M(f) = f|_N$.

LEMMA 8. *Le $M \subset \mathbb{R}$ be a set with property (D), and $I := (\inf M, \sup M)$. Let Q be an n-dimensional linear space of real-valued right-continuous functions defined on I. For $f \in Q$ assume: f is bounded on every compact subinterval of I; f has no strong alternations of length greater n; on each of the intervals of $I \backslash \overline{M}$ f is equal to its value at the left endpoint of the interval. Let the restriction $E_M{}^I(Q)$ of Q to M be an n-dimensional (oriented) T-space. Then if we define $W := \{h: I \to \mathbb{R} \mid$ there are $f \in Q$, $a \in I$, $\alpha \in \mathbb{R}$ such that for $x \in I$ $h(x) = \int_a^x f(t)\,dt + \alpha\}$, $E_M{}^I(W)$ is an $(n+1)$-dimensional oriented T-space on M.*

Proof. (1) Suppose there is an $h \in W \backslash \{0\}$ with zeros $t_1, \ldots, t_{n+1} \in M$, $t_1 < \cdots < t_{n+1}$. Then there exist $f \in Q$, $a \in M$ and $\alpha \in \mathbb{R}$ such that

$$\int_a^{t_i} f(t)\,dt = -\alpha \qquad \text{for} \quad i = 1, \ldots, n+1.$$

and, consequently,

$$\int_{t_i}^{t_{i+1}} f(t)\,dt = 0 \qquad \text{for} \quad i = 1, \ldots, n.$$

As $E_M{}^I(Q)$ is a T-space on M, and M has property (D), f cannot identically vanish on (t_i, t_{i+1}), and so f has a strong alternation of length greater 1 on (t_i, t_{i+1}). Thus, f has a strong alternation of length greater n on M contradicting the hypothesis.

(2) Suppose there are an $h \in W$ and $t_1, \ldots, t_{n+2} \in M$ with $t_1 < \cdots < t_{n+2}$ and $\alpha_i := (-1)^i h(t_i) > 0$ for $i = 1, \ldots, n+2$. Then there exist $f \in Q$, $a \in M$ and $\alpha \in \mathbb{R}$ such that

$$\int_a^{t_i} f(t)\,dt = (-1)^i \alpha_i - \alpha \qquad \text{for} \quad i = 1, \ldots, n+2,$$

and, consequently,

$$\int_{t_i}^{t_{i+1}} f(t)\,dt = (-1)^{i+1} (\alpha_i + \alpha_{i+1}) \qquad \text{for} \quad i = 1, \ldots, n+1.$$

So $\int_{t_i}^{t_{i+1}} f(t)\,dt$ is alternatingly positive and negative for $i = 1, \ldots, n+1$, and f has a strong alternation of length $n+1$ contradicting the hypothesis.

LEMMA 8′. *Let $M \subset \mathbb{R}$ be a set with property (D), and $I :=$* (inf M, sup M).

For $i = 1,\dots, n$ let Q_i be i-dimensional linear spaces of real-valued right-continuous functions defined on I. For $f \in Q_i$ assume: f is bounded on every compact subinterval of I; f has no strong alternations of length greater i; on each of the intervals of $I \backslash \overline{M}$ f is equal to its value at the left endpoint of the interval. Let Q_1 consist of the constant functions, assume $Q_1 \subset \cdots \subset Q_n$, and let $E_M{}^I(Q_1) \subset \cdots \subset E_M{}^I(Q_n)$ be a (normed) chain of (oriented) T-spaces on M. If we then define $W_1 := Q_1$, $W_{i+1} := \{h: I \to \mathbb{R} \mid$ there are $f \in Q_i$, $a \in I$, $\alpha \in \mathbb{R}$ such that for $x \in I$ $h(x) = \int_a^x f(t)\,dt + \alpha\}$ for $i = 1,\dots, n$, $E_M{}^I(W_1) \subset \cdots \subset E_M{}^I(W_{n+1})$ is a normed chain of oriented T-spaces on M.

THEOREM 3. *If hypothesis (A) is fulfilled and M has property (D), there is a function strongly adjoined to U_n.*

Proof. Because of statement (3) at the beginning of this section we may assume without loss of generality that the hypotheses of Lemma 5 are fulfilled.

$n = 2$. The quadratic polynomials, restricted to M, form an oriented T-space containing U_2.

$n - 1 \Rightarrow n$. As described in Lemma 5, every $f \in U_n$ can be extended to a function $\bar{f}$ defined on $I := (\inf M, \sup M)$ with the properties shown in Lemma 6 and 7. For $i = 1,\dots, n - 1$ let $Q_i := \{\bar{f}_+' \mid \bar{f} \in \overline{U}_n\}$. Then $E_M{}^I(Q_1) \subset \cdots \subset E_M{}^I(Q_{n-1})$ is a normed chain of oriented T-spaces since we have $E_M{}^I(Q_i) = D_+U_{i+1}$ for $i = 1,\dots, n - 1$ (see Theorem 2). By induction hypothesis there is a function $w: M \to \mathbb{R}$ strongly adjoined to $E_M{}^I(Q_{n-1})$. We make the additional induction hypothesis that w is continuous on M, and extend w to a function g defined on all of I in the following way: Because of Lemmas 3 and 4 w can be continuously extended to a function $\bar{w}$ defined on $I \cap \overline{M}$. For $t \in I \cap \overline{M}$, we set $g(t) = \bar{w}(t)$. If T is a connected subset of $I \backslash \overline{M}$, on T we set g constantly equal to its value in the left endpoint of T. g is bounded on every compact subinterval of I, and I can be split into $k \leqslant n - 1$ subintervals A_k such that g is alternatingly increasing and decreasing (not necessarily in the strict sense) on $A_1,\dots, A_k$. It is easy to see that then the last two statements hold for all functions in $Q_n := \operatorname{span} Q_{n-1} \cup \{g\}$. Now we can apply Lemma 8′: The absolute continuity of the functions $\bar{f} \in \overline{U}_n$ yields

$$\int_a^x \bar{f}_+'(t)\,dt = f(x) - f(a) \qquad \text{for all} \quad a, x \in I,$$

and we get $E_M{}^I(W_i) = U_i$ for $i = 1,\dots, n$. Setting $h(x) := \int_a^x g(t)\,dt$ for some fixed $a \in M$, $h|_M$ is a function continuous on M and strongly adjoined to U_n.

With the remarks at the beginning of paragraph 1 we can formulate our main result.

THEOREM 3′. *If M is a totally ordered set with property (D) and U_n is an n-dimensional oriented T-space on M with $n \geqslant 2$, there is a function strongly adjoined to U_n.*

The following statement is also immediate from the proof of Theorem 3 together with statements (1) and (2) at the beginning of this Section.

THEOREM 4. *If M is a real open interval and $U_n \subset C(M)$ is an n-dimensional T-space there is a function $f \in C(M)$ adjoined to U_n.*

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